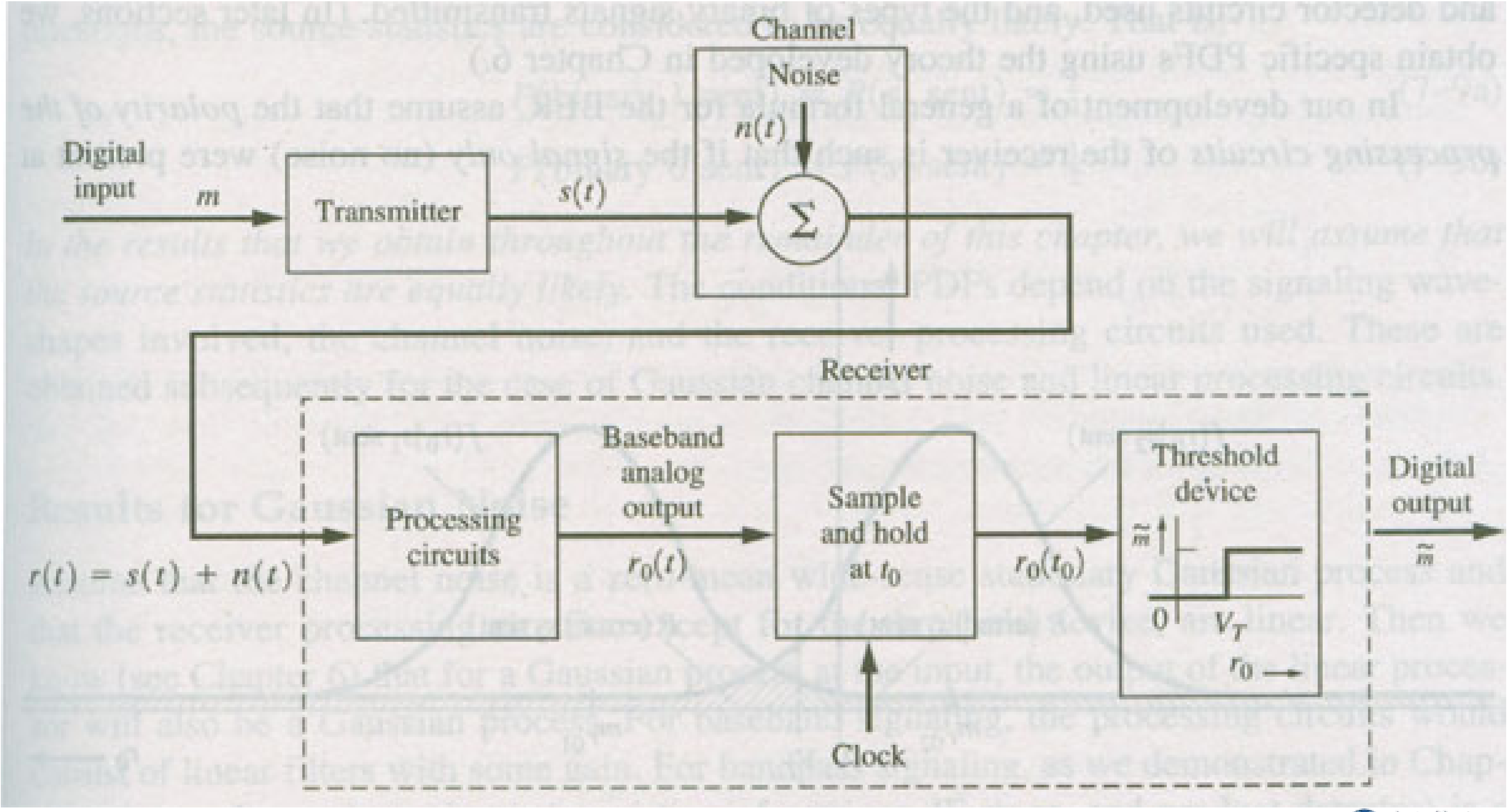
Chapter 7

Performance of communication systems corrupted by noise

7.1 Error probability for binary signaling

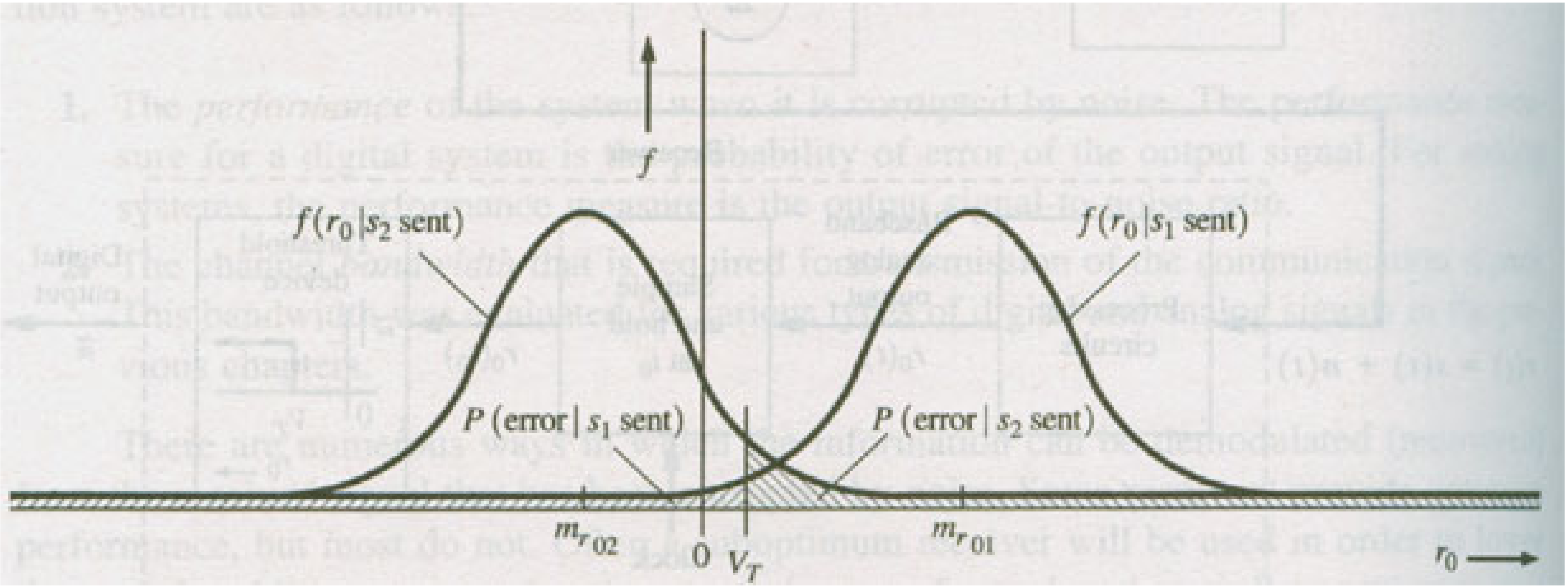
General results

General binary communication system



General results

The PDFs for the two random variables $r_0=r_{01}$ and $r_0=r_{02}$ are conditional PDFs, since they depend, respectively, on a binary 1 or a binary 0 being transmitted.



General results

Assume

When a binary 1 is sent, If the signal only (no noise) were present at the reciever input, $r_0 > V_T$.

When a binary 0 is sent, $r_0 < V_T$.

When signal plus noise is present at the receiver input, errors can occur in two ways:

An error occur when $r_0 < V_T$ if a binary 1 is sent:

$$P(error \mid s_1\_sent) = \int_{-\infty}^{V_t} f(r_0 \mid s_1) dr_0$$

Similarly, an error occur when $r_0 > V_T$ if a binary 0 is sent:

$$P(error \mid s_2\_sent) = \int_{-\infty}^{V_t} f(r_0 \mid s_2) dr_0$$

General results

The BER is then

$$P_e = P(error \mid s_1\ sent)P(s_1\ sent) + P(error \mid s_2\ sent)P(s_2\ sent)$$

The general expression for BER of any binary communication system is:

$$P_e = P(s_1\ sent)\int_{-\infty}^{V_T} f(r_0 \mid s_1)dr_0 + P(s_2\ sent)\int_{V_T}^{\infty} f(r_0 \mid s_2)dr_0$$

Results for Gaussian noise

Assume

The channel noise is a zero-mean wide-sense stationary Gaussian process, the receiver processing circuits, except for the threshold device, are linear.

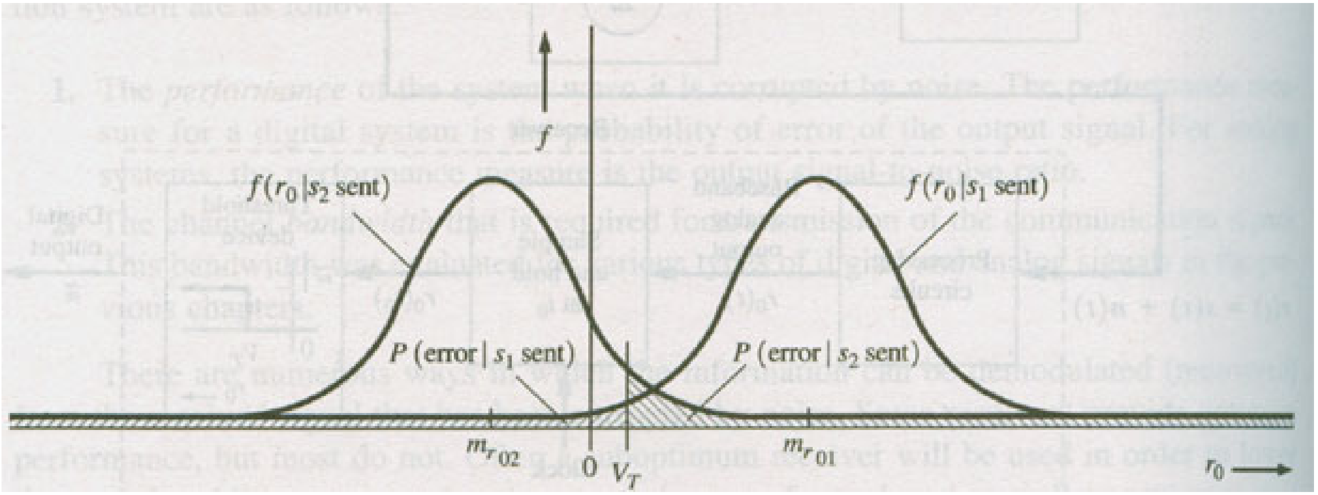
Since the output sample n_0 is a zero-mean Gaussian random variable, the total output sample r_0 is a Gaussian random variable with a mean value of either s_{01} or s_{02}.

The mean value of r_0 is:

$m_{r01}=s_{01}$, **when a binary 1 is sent**

$m_{r02}=s_{02}$, **when a binary 0 is sent**

Results for Gaussian noise



Thus, two conditional PDFs are

$$f(r_0 \mid s_2) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(r_0 - s_{02})^2/(2\sigma_0^2)}$$

$$f(r_0 \mid s_1) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(r_0 - s_{01})^2/(2\sigma_0^2)}$$

Using equally likely source statistics, the BER becomes:

$$p_e = \frac{1}{2}\int_{-\infty}^{V_T} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(r_0 - s_{01})^2/(2\sigma_0^2)} dr_0 + \frac{1}{2}\int_{V_T}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(r_0 - s_{02})^2/(2\sigma_0^2)} dr_0$$

Results for Gaussian noise

$$p_e = \frac{1}{2}\int_{-(V_T - s_{01})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} d\lambda + \frac{1}{2}\int_{(V_T - s_{02})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} d\lambda$$

or

$$p_e = \frac{1}{2} Q\left(\frac{(-V_T + s_{01})}{\sigma_0}\right) + \frac{1}{2} Q\left(\frac{(V_T - s_{02})}{\sigma_0}\right)$$

For binary signaling in Gaussian noise and with the optimum threshold $V_T = \frac{s_{01} + s_{02}}{2}$ **, the BER is**

$$p_e = Q\left(\frac{(s_{01} - s_{02})}{2\sigma_0}\right) = Q\left(\sqrt{\frac{(s_{01} - s_{02})^2}{4\sigma_0^2}}\right)$$

Results for matched-filter

To minimize p_e, we need to find the linear filter that maximizes

$$\frac{[s_{01}(t_0)-s_{02}(t_0)]^2}{\sigma_0^2}=\frac{[s_d(t_0)]^2}{\sigma_0^2}$$

where

$s_d(t_0)=s_{01}(t_0)-s_{02}(t_0)$ is the difference signal sample value.

$\sigma_0^2=\overline{n_0^2(t)}$ is average output noise power.

The impulse response of the matched filter for binary signaling is

$$h(t)=C\left[s_1(t_0-t)-s_2(t_0-t)\right]$$

Results for matched-filter

The output peak signal to average noise ratio that is obtained from the matched filter is

$$\frac{[s_d(t_0)]^2}{\sigma_0^2} = \frac{2E_d}{N_0}$$

Thus, for binary signaling corrupted by **white Gaussian noise**, **matched-filter reception**, and by using **optimum threshold setting**, the BER is

$$p_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

Results for colored Gaussian noise

For colored Gaussian noise

A prewhitening filter is insert ahead of the receiver processing circuits.

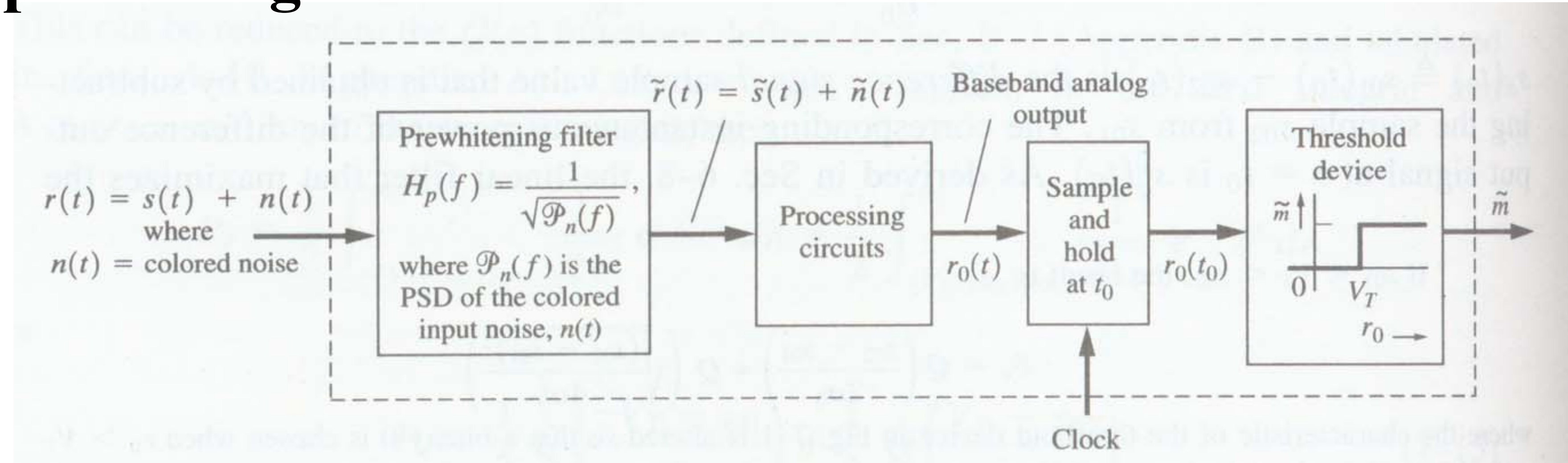


Figure 7–3 Matched-filter receiver for colored noise.

The transfer function of the prewhitening filter is

$$H_p(f) = \frac{1}{\sqrt{p_n(f)}}$$

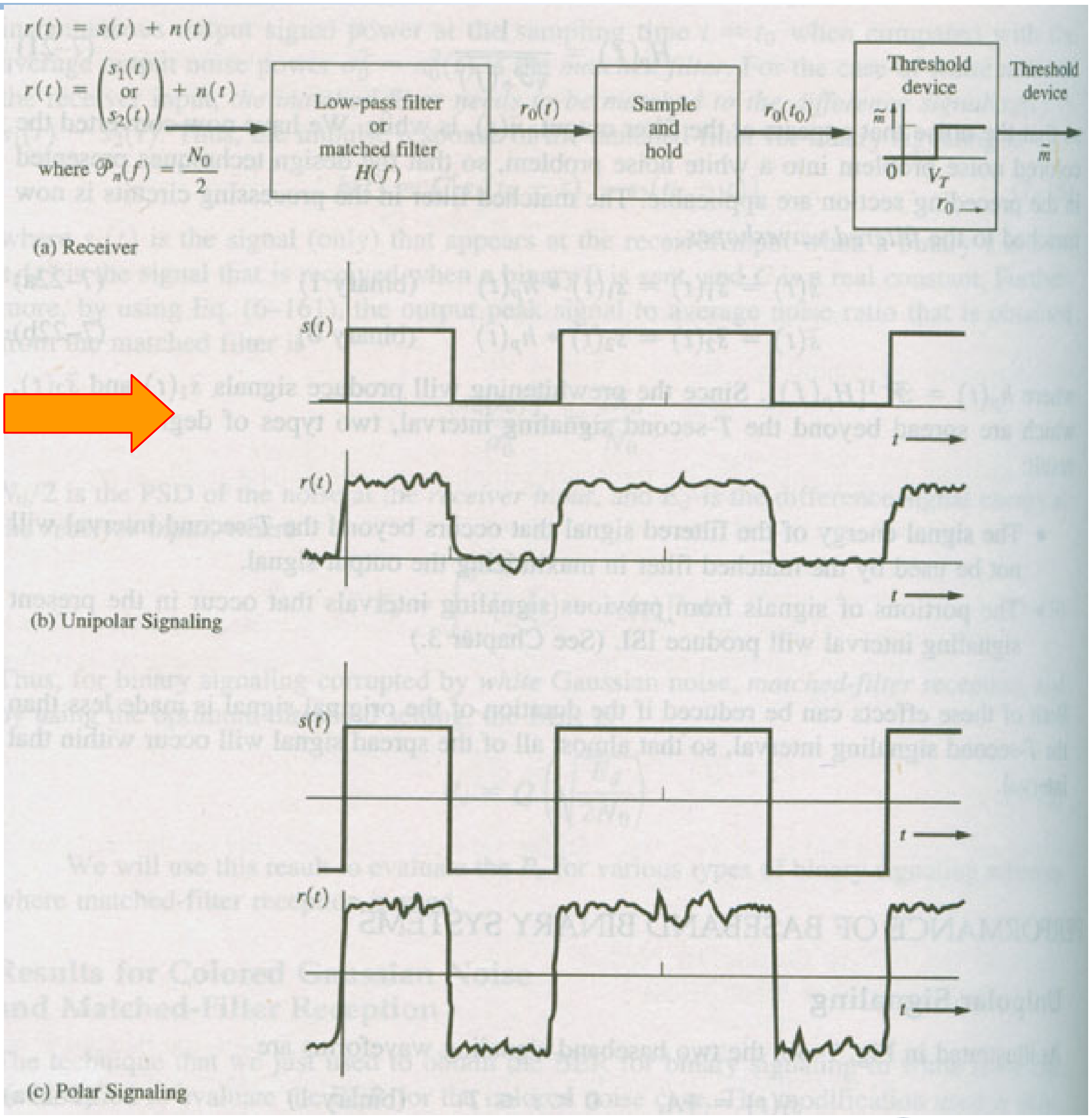
7.2 performance of baseband binary systems

Unipolar signaling

The **two baseband signaling waveform** are:

$$s_1(t) = +A \quad \text{(binary 1)}$$

$$s_2(t) = 0 \quad \text{(binary 0)}$$



Unipolar signaling

Receiver that uses an LPF

- The equivalent bandwidth of the LPF is B >2/T.
- $s_{01}(t_0) \approx A$, $s_{02}(t_0) \approx 0$.
- The noise power at the output of the filter is
- Decision with $V_T = A/2$

$$\sigma_0^2 = (N_0/2)(2B)$$

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2}{4N_0B}}\right)$$

(low-pass filter)

Unipolar signaling

Receiver that uses a matched-filter

- The sampling time is $t_0 = T$.
- The energy in the difference signal is $E_d = A^2 T$.
- Decision with $V_T = AT/2$

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

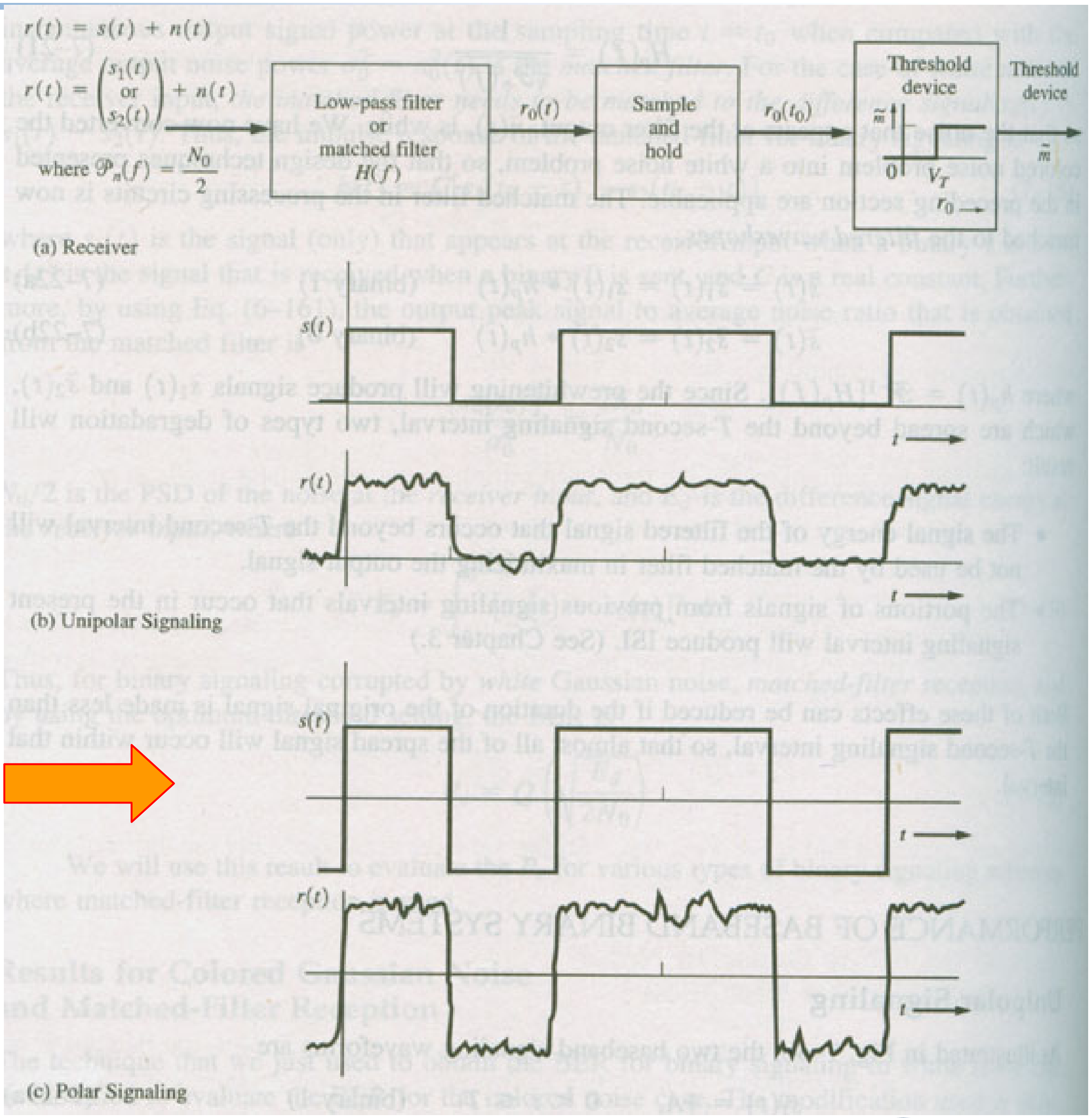
(matched filter)

Where the average energy per bit is $E_b = A^2 T/2$

Polar signaling

The **baseband polar signaling waveform** is:

$$s_1(t) = +A \qquad \text{(binary 1)}$$
$$s_2(t) = -A \qquad \text{(binary 0)}$$



Polar signaling

Receiver that uses an LPF

- The equivalent bandwidth of the LPF is $B \geqslant 2/T$.
- $s_{01}(t_0) \approx A$, $s_{02}(t_0) \approx -A$.
- The noise power at the output of the filter is $\sigma_0^2 = N_0 B$
- Decision with $V_T=0$

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2}{N_0 B}}\right)$$

(low-pass filter)

Polar signaling

Receiver that uses a matched-filter

- The sampling time is $t_0=T$.
- The energy in the difference signal is

$$E_d = \int_0^T [s_1(t)-s_2(t)]^2\,dt = (2A)^2 T$$

- Decision with $V_T=0$

The BER is

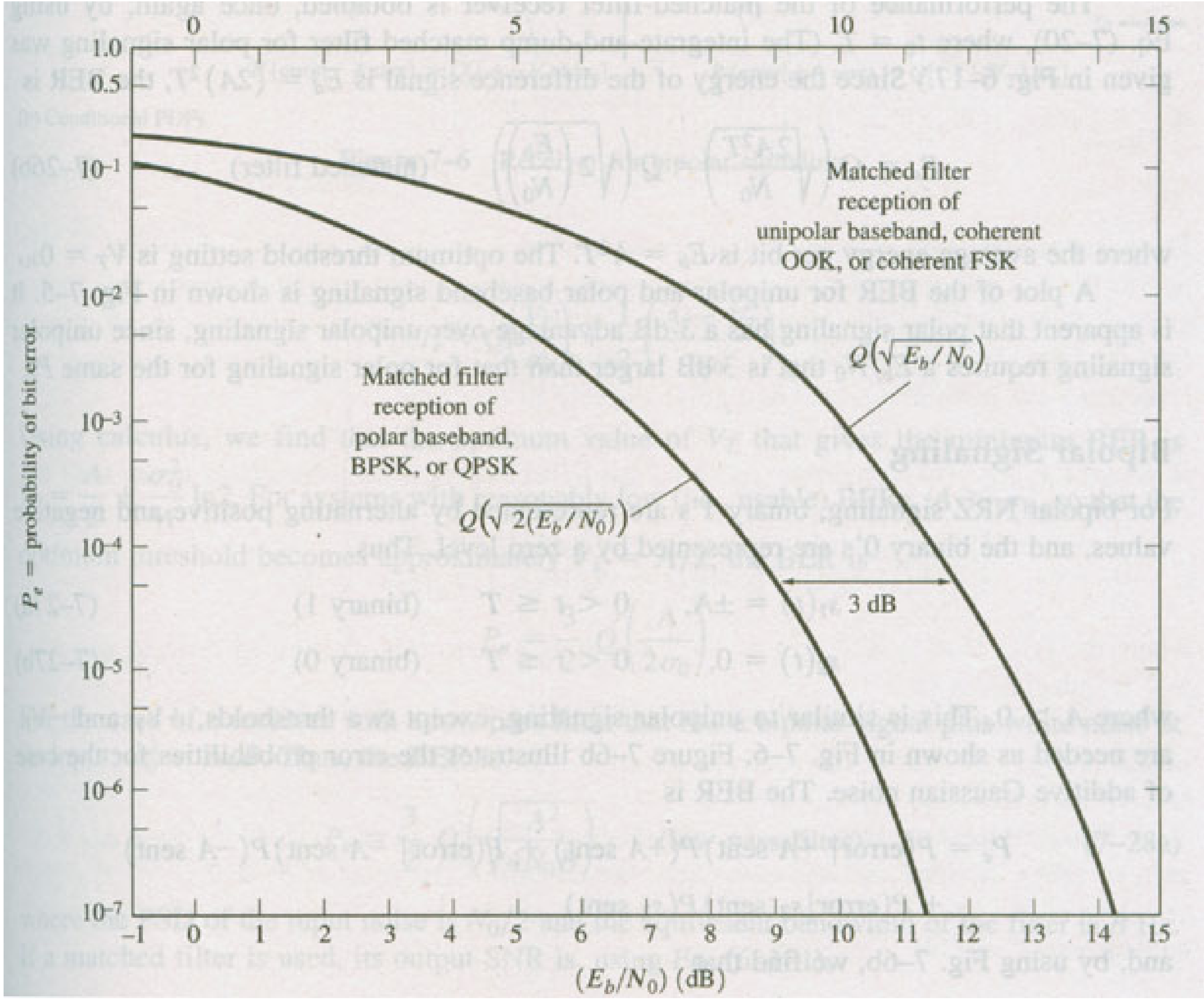
$$P_e = Q\left(\sqrt{\frac{2A^2T}{2N_0}}\right) = Q\left(\sqrt{2\left(\frac{E_b}{N_0}\right)}\right)$$

(matched filter)

Where the average energy per bit is $E_b=A^2T$

Polar & Unipolar signaling

Comparison between polar signaling and unipolar signaling

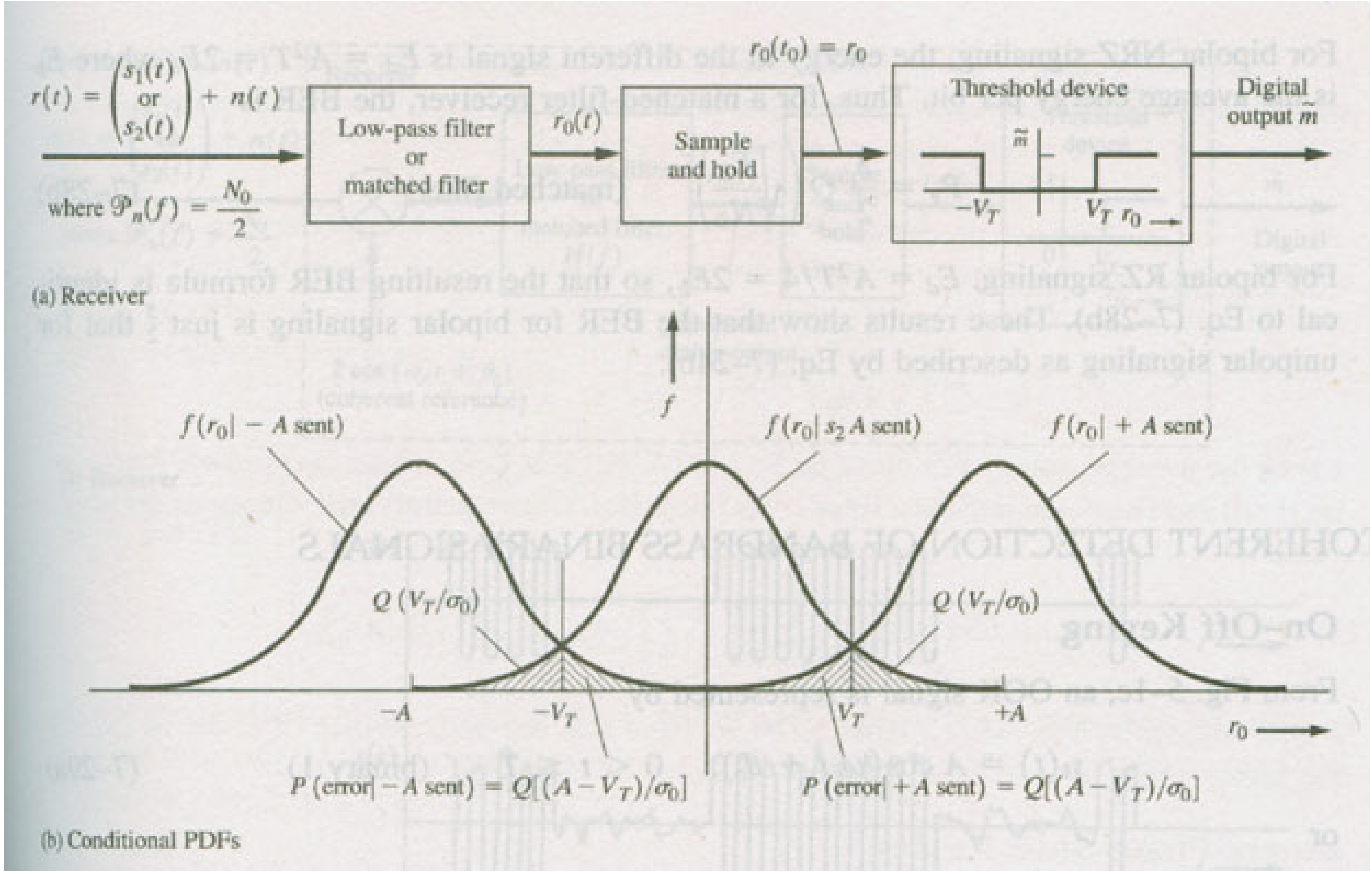


The polar signaling has a 3-dB advantage over unipolar signaling

Bipolar signaling

The **bipolar NRZ signaling** is:

$s_1(t) = \pm A$ **(binary 1) ,** $s_2(t) = 0$ **(binary 0)**



Bipolar signaling

The optimum value of V_T is approximately $V_T=A/2$.

The BER is

$$P_e = \frac{3}{2} Q\left(\frac{A}{2\sigma_0}\right) = \frac{3}{2} Q\left(\sqrt{\frac{A^2}{4\sigma_0^2}}\right)$$

Receiver that uses a low-pass filter

If the receiver has a bipolar signal plus white noise at its input, $\sigma_0^2 = N_0 B$

The BER is:

$$P_e = \frac{3}{2} Q\left(\frac{A^2}{4N_0 B}\right)$$

(low-pass filter)

Bipolar signaling

Receiver that uses a matched-filter

- Matched filter **output *SNR*** is $\left(\frac{S}{N}\right)_{out} = \frac{A^2}{\sigma_0^2} = \frac{2E_d}{N_0}$
- The **energy in the difference signal** is $E_d = A^2 T = 2E_b$

The BER is

$$P_e = \frac{3}{2} Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

(matched filter)

Note:

These results show that the BER for bipolar signaling is just 3/2 that for unipolar signaling.

7.3 coherent detection of bandpass binary signals

OOK

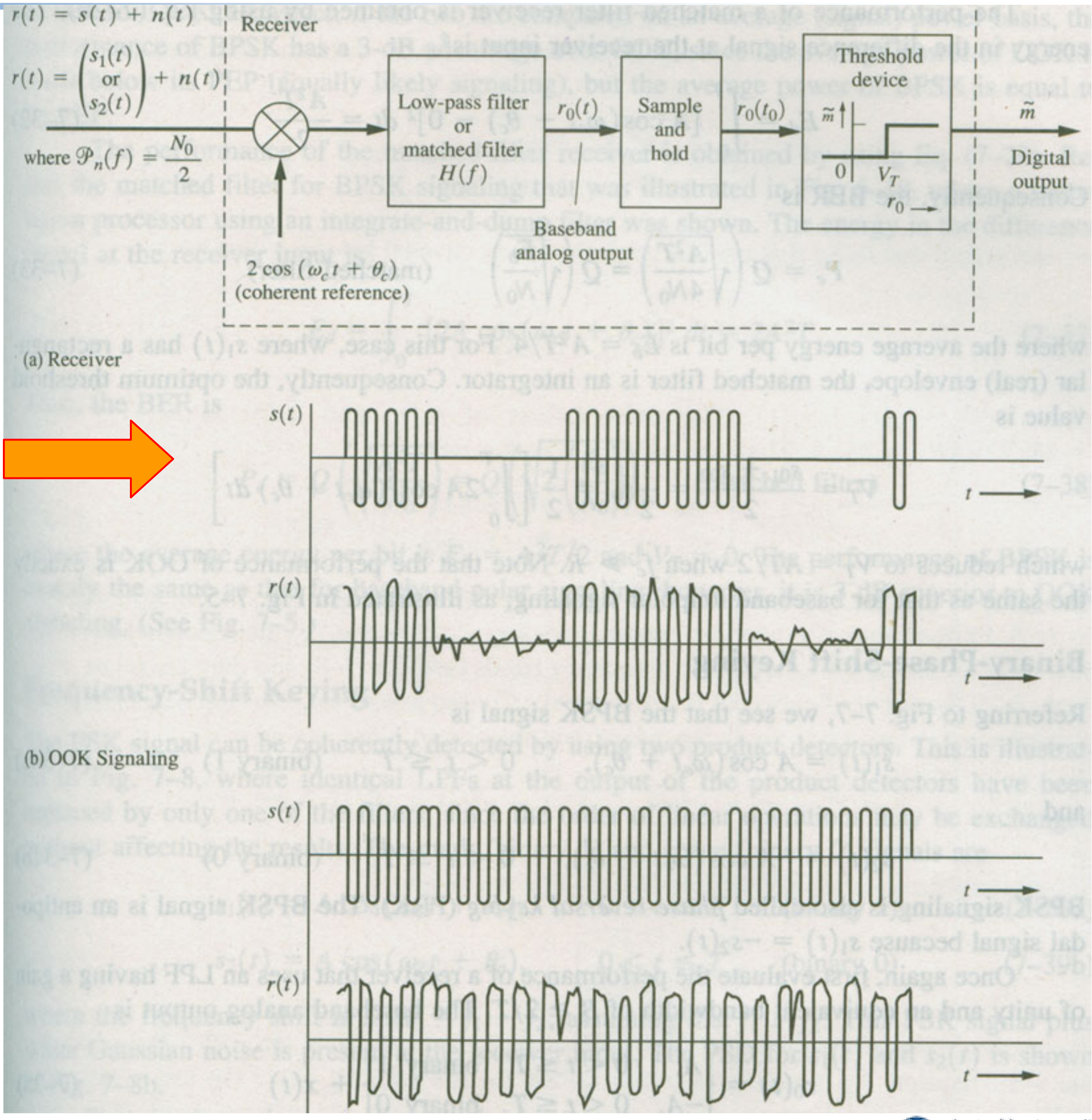
OOK signal is

$$s_1(t) = A\cos(\omega_c t + \theta_c)$$

(Binary 1)

$$s_2(t) = 0$$

(Binary 0)



OOK

Receiver that uses an LPF

- **The equivalent bandwidth of the LPF is $B \geqslant 2/T$.**
- **the baseband analog output is**

$$r_0(t) = \begin{cases} A, \ binary \ \ 1 \\ 0, \ binary \ \ 0 \end{cases} + x(t)$$

- **The noise power at the output of the filter is** $\sigma_0^2 = 2N_0B$
- **Decision with $V_T = A/2$**

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2}{8N_0B}}\right)$$

(narrowband filter)

Receiver that uses a matched-filter

- The energy in the difference signal is

$$E_d = \int_0^T [A\cos(\omega_c t + \theta_c) - 0]^2 = A^2 T / 2$$

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2 T}{4N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

(matched filter)

where the average energy per bit is $E_b = A^2T/4$

The optimum threshold value is

$$V_T = \frac{s_{01} + s_{02}}{2} = \frac{1}{2} s_{01} = \frac{1}{2}\int_0^T 2A\cos^2(\omega_c t + \theta_c)dt$$

when $f_c >> R$, $V_T = AT/2$

BPSK

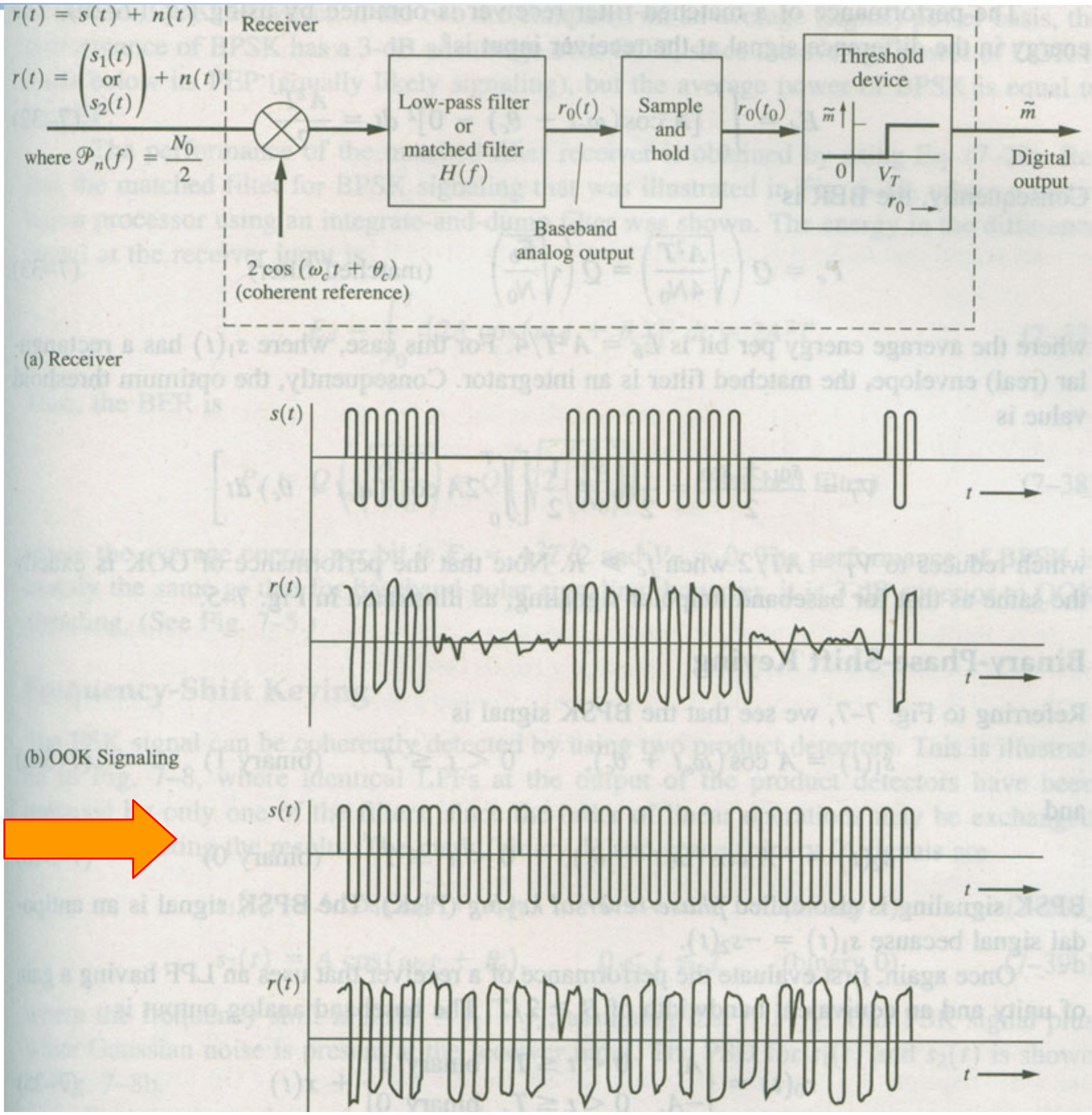
BPSK signal is

$$s_1(t) = A\cos(\omega_c t + \theta_c)$$

(Binary 1)

$$s_2(t) = -A\cos(\omega_c t + \theta_c)$$

(Binary 0)



Receiver that uses an LPF

- The equivalent bandwidth of the LPF is $B \geqslant 2/T$.
- the baseband analog output is

$$r_0(t) = \begin{cases} A, & binary\ 1 \\ -A, & binary\ 0 \end{cases} + x(t)$$

- The noise power at the output of the filter is $\sigma_0^2 = 2N_0 B$
- Decision with $V_T = 0$

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2}{2N_0 B}}\right)$$

(narrowband filter)

Comparison beteewn BPSK and OOK for a given value of N_0.

- On a PEP basis , 6 *dB* less signal power is required for BPSK to give the same p_e as that for OOK.

- if the two compared on an average power basis, the performance of BPSK has a 3-*dB* advantage over OOK.

Receiver that uses a matched-filter

- **The energy in the difference signal is**

$$E_d = \int_0^T [2A\cos(\omega_c t + \theta_c)]^2 = 2A^2T$$

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2T}{N_0}}\right) = Q\left(\sqrt{2\left(\frac{E_b}{N_0}\right)}\right)$$

(matched filter)

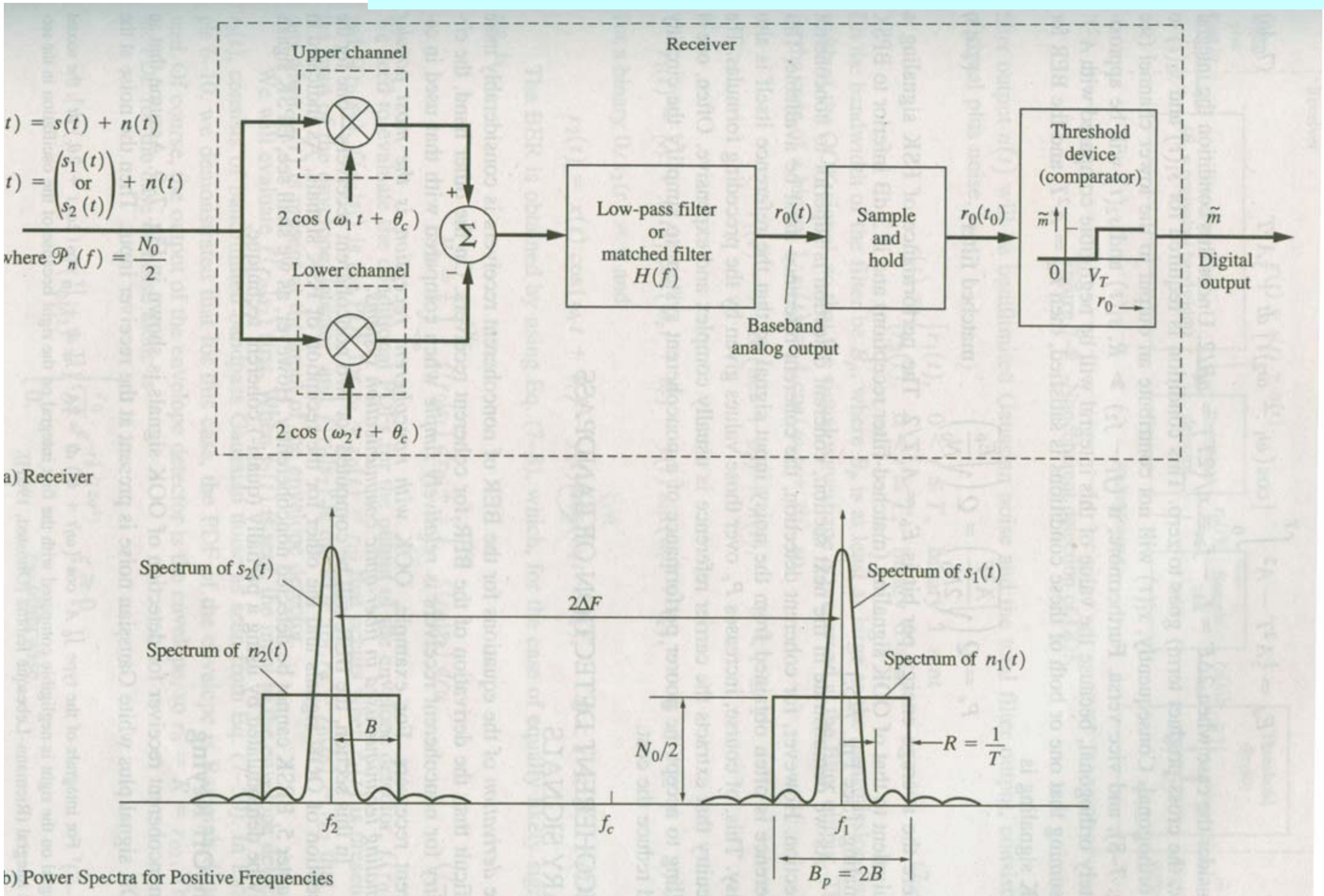
where the average energy per bit is $E_b = A^2T/2$

The optimum threshold value is $V_T = 0$

FSK

The FSK signal:

$$s(t)=\begin{cases} s_1(t)=A\cos(\omega_1 t+\theta_1), & data=1 \\ s_2(t)=A\cos(\omega_2 t+\theta_2), & data=0 \end{cases}$$



Receiver that uses an LPF

- The equivalent bandwidth of the LPF is $2/T \leqslant B < \Delta F$.

The LPF, when combined with the frequency translation produce by the product detectors, act as dual bandpass filters:

One centered at $f = f_1$ and the other at $f = f_2$. each has an equivalent bandwidth of $B_p = 2B$.

The input noise consists of two narrowband components $n_1(t)$ and $n_2(t)$:

$$n_1(t) = x_1(t)\cos(\omega_1 t + \theta_c) - y_1(t)\sin(\omega_2 t + \theta_c)$$

$$n_2(t) = x_2(t)\cos(\omega_2 t + \theta_c) - y_2(t)\sin(\omega_2 t + \theta_c)$$

Receiver that uses an LPF

The baseband analog output is:

$$r_0(t) = \begin{cases} +A, & 0 < t \le T,\ data = 1 \\ -A, & 0 < t \le T,\ data = 0 \end{cases} + n_0(t)$$

Where s_{01}=+A, s_{02}=-A , $n_0(t) = x_1(t) - x_2(t)$

The optimum threshold setting is V_T= 0.

The output noise power is:

$$\overline{n_0^2(t)} = \sigma_0^2 = \overline{x_1^2(t)} + \overline{x_2^2(t)} = \overline{n_1^2(t)} + \overline{n_2^2(t)} = 4N_0B$$

The BER is

$$P_e = Q\left(\sqrt{\frac{A^2}{4N_0B}}\right)$$

FSK

Receiver that uses a matched-filter

The energy in the difference signal is:

$$E_d = \int_0^T [A\cos(\omega_1 t + \theta_c) - A\cos(\omega_2 t + \theta_c)]^2 dt$$

$$= A^2T - A^2 \int_0^T [\cos(\omega_1 - \omega_2)t]dt$$

Assume $2\Delta F = f_1 - f_2 = n/(2T) = nR/2$, or $f_1 - f_2 >> R$, or both of these conditions is satisfied, then:

$$E_d = A^2T$$

The BER for FSK signaling is:

$$P_e = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

where $E_b = A^2T/2$

FSK

The performance of FSK signaling is equivalent to that of OOK signaling and is 3 dB inferior to BPSK signaling.